Variational Principle and Existence Theorems of Periodic Solutions for a Class of Damped Vibration Problems

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Abstract: In the present paper, we research the following damped vibration problem

\[
\begin{align*}
\ddot{u}(t) + g(t) \dot{u}(t) &= \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\
u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0
\end{align*}
\]

where \( T > 0 \), \( g \in L^1(0, T; \mathbb{R}) \), \( G(t) = \int_0^t g(s) \, ds \) and \( F: [0, T] \times \mathbb{R}^n \to \mathbb{R} \). The variational principle is given, and two existence theorems of periodic solutions are obtained.

Key words: critical point; periodic solution; second order Hamiltonian system; Sobolev's inequality

1 Introduction

Consider the following damped vibration problem

\[
\begin{align*}
\ddot{u}(t) + g(t) \dot{u}(t) &= \nabla F(t, u(t)), \quad \text{a.e. } t \in [0, T], \\
\dot{u}(0) - \dot{u}(T) &= u(0) - u(T) = 0
\end{align*}
\]

(1.1)

where \( T > 0 \), \( g \in L^1(0, T; \mathbb{R}) \), \( G(t) = \int_0^t g(s) \, ds \) and \( F: [0, T] \times \mathbb{R}^n \to \mathbb{R} \). Satisfies the following assumption:

(A) \( F(t, x) \) is measurable in \( t \) for every \( x \in \mathbb{R}^n \)

and continuously differentiable in \( x \) for a.e. \( t \in [0, T] \) and there exist \( a \in C(R^n; R^n); \ b \in L_1(0; T; \mathbb{R}^n) \).
We have the following facts.

**Theorem 2.1** The functional $\Psi$ is continuously differentiable and weakly lower semicontinuous on $H_T$.

**Proof** Let $\mathcal{L}(t; x; y) = e^{G(t) \frac{t^2}{2}} + G(t) F(t, x)$ for all $t, x \in \mathbb{R}^n$ and $\forall t \in [0, T]$. Then by Theorem 1.4 in [2] the functional $\Psi$ is continuously differentiable on $H_T$ and

$$
\Psi(u), v = \int_0^T e^{G(t)} \left( \dot{u}(t), \dot{v}(t) \right) dt + \int_0^T e^{G(t)} \left( \nabla F(t, u(t)), v(t) \right) dt
$$

for all $u, v \in H_T$. Moreover, the proof for the weakly lower semicontinuity of $\Psi$ is similar to the corresponding parts in [3].

**Theorem 2.2** If $u \in H_T$ is a solution of Euler equation $\Psi'(u) = 0$, then $u$ is a solution of problem (1.1).

**Proof** Since $\Psi'(u) = 0$,

$$
0 = \Psi(u), v = \int_0^T e^{G(t)} \left( \dot{u}(t), \dot{v}(t) \right) dt + \int_0^T e^{G(t)} \left( \nabla F(t, u(t)), v(t) \right) dt
$$

for all $v \in H_T$, i.e.

$$
\int_0^T e^{G(t)} \left( \dot{u}(t), \dot{v}(t) \right) dt = - \int_0^T e^{G(t)} \left( \nabla F(t, u(t)), v(t) \right) dt
$$

for all $v \in H_T$. By Fundamental Lemma and Remarks 1 in [13], in $L^1$ we know that $e^{G(t)} \dot{u}(t)$ has a weak derivative, and

$$
\int e^{G(t)} \dot{u}(t) = e^{G(t)} \left( \nabla F(t, u(t)) \right)
$$

ae on $\in [0, T]$.

(2.1)

$$
\int e^{G(t)} \dot{u}(t) \int e^{G(s)} \left( \nabla F(s, u(s)) \right) ds + c
$$

ae on $\in [0, T]$.

(2.2)

$$
\int_0^T e^{G(t)} \nabla F(t, u(t)) dt = 0
$$

(2.3)

where $c$ is a constant. We identify the equivalence class $e^{G(t)} \dot{u}(t)$ and its continuous representative

$$
\int_0^T e^{G(t)} \nabla F(s, u(s)) ds + c
$$

Then, by (2.2), (2.3) and the existence of $u$, one has

$$
\dot{u}(0) - \dot{u}(T) = u(0) - u(T) = 0
$$

Moreover, by (2.1) we know

$$
\dot{u}(t) + g(t) \dot{u}(t) = \nabla F(t, u(t))
$$
a.e. $\in [0, T]$.
Hence $u$ is a solution of problem (1.1). This completes the proof.

3 Existence of Periodic Solutions

Theorem 3.1 Assume following hold:

$(F_1)$ $\lim\inf_{t \to \infty} F(t, x) \geq 0$, a.e. $t \in [0, T]$, and

$(F_1')$ whenever $|u_\varepsilon| \subseteq H_r^1$ is such that $\|u_\varepsilon\| \to \infty$
and $\frac{u_\varepsilon}{\|u_\varepsilon\|} \left( \int_0^T e^{g(t)} b(t) dt \right)^{-\frac{1}{2}} \to 1$,

$$\lim_{n \to \infty} \sup \int_0^T e^{g(t)} F(t, u(t)) dt = + \infty,$$

where $u_\varepsilon = \frac{1}{T} \int_0^T u_\varepsilon(t) dt$. Then the problem (1.1) has at least one solution which minimizes $\varphi$ on $H^1_r$.

Proof If there are a sequence $\{u_n\}$ and a constant $c$ such that $\|u_n\| \to \infty$ (as $m \to \infty$) and $\varphi(u_n) \leq c$, $n = 1, 2, \ldots$, then let $u_\varepsilon = \frac{u_n}{\|u_n\|}$. Since $H^1_r$ is a Hilbert space, there is a point $v_0 \in H^1_r$ and a subsequence of $\{v_n\}$, we still note by $\{v_n\}$, such that $v_\varepsilon \to v_0$ in $H^1_r$.

For any $\varepsilon > 0$, by $(F_1)$ there is a $M > 0$ such that $F(t, x) > -\frac{\varepsilon}{2} |x|^2$ for all $x \in \mathbb{R}^N$ with $|x| > M$ and a.e. $t \in [0, T]$. Let $aM = \max |x| \leq M \varphi(|x|)$. Then by the assumption (A), one has $|F(t, x)| \leq aM$ for all $x \in \mathbb{R}^N$ with $|x| \leq M$ and a.e. $t \in [0, T]$. Hence

$$F(t, x) \geq -\frac{\varepsilon}{2} |x|^2 - aM^2,$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. Consequently,

$$\frac{c}{\|u_\varepsilon\|^2} > \frac{\varphi(u_\varepsilon)}{\|u_\varepsilon\|^2} =$$

$$\frac{1}{2} \int_0^T e^{g(t)} |v_\varepsilon(t)|^2 dt + \frac{c}{\|u_\varepsilon\|^2} \int_0^T e^{g(t)} F(t, u_\varepsilon(t)) dt \geq$$

$$\frac{1}{2} \int_0^T e^{g(t)} |v_\varepsilon(t)|^2 dt - \frac{c}{\|u_\varepsilon\|^2} \int_0^T e^{g(t)} \left[ \frac{c}{2} |u_\varepsilon(t)|^2 + aM^2 \right] dt =$$

$$\frac{1}{2} \int_0^T e^{g(t)} |v_\varepsilon(t)|^2 dt - \frac{c}{2} \int_0^T e^{g(t)} |v_\varepsilon(t)|^2 dt,$$

It implies $\int_0^T e^{g(t)} |v_\varepsilon(t)|^2 dt \geq 1$. On the other hand, by weakly lower semi-continuity of the norm, one has $\|u_\varepsilon\| \leq \lim\inf \|v_\varepsilon\| = 1$.

Hence $\|v_\varepsilon(t)\| = 0$ a.e. $t \in [0, T]$, so that $|v_\varepsilon(t)| = 1$ constant for a.e. $t \in [0, T]$, and hence $|v_\varepsilon|^2 = 1$ for a.e. $t \in [0, T]$.

Consequently, $\frac{\|u_\varepsilon(t)\|}{\|u_\varepsilon\|} \left( \int_0^T e^{g(t)} dt \right)^{\frac{1}{2}} \to 1$, and hence $|u_\varepsilon| \to + \infty$. By (F2),

$$\lim_{n \to \infty} \sup \int_0^T e^{g(t)} F(t, u_\varepsilon(t)) dt = + \infty,$$

Hence

$$c = \lim\sup_{n \to \infty} \varphi(u_n) \geq \lim\sup_{n \to \infty} \int_0^T e^{g(t)} F(t, u_\varepsilon(t)) dt = + \infty,$$

This is a contradiction. Hence $\varphi$ is coercive on $H^1_r$. By Theorem 2.1 we know that $\varphi$ is weak lower semi-continuity, and hence $\varphi$ is bounded below and has a bounded minimizing sequence. Therefore, $\varphi$ has a minimum on $H^1_r$ by Theorem 1.1 in [3]. Consequently, the conclusion follows from Theorem 2.2. This completes the proof.

Remark 3.1 There are functions satisfying Theorem 3.1. For example,

$$F(t, x) = \begin{cases} \frac{\omega^2}{8} |x| - \frac{1}{6} |x|^2, & |x| \geq 1, \\ -\frac{1}{4} \omega^2 |x|^2 + \frac{3}{8} \omega^2 |x|^2 - \frac{1}{6} \omega^2 |x|^2 |x|, & |x| \leq 1. \end{cases}$$

Theorem 3.2 If the following hold:

$(F_3)$ whenever $|u_\varepsilon| \subseteq H^1_r$ is such that $\|u_\varepsilon\| \to \infty$
and $\frac{u_\varepsilon}{\|u_\varepsilon\|} \left( \int_0^T e^{g(t)} dt \right)^{-\frac{1}{2}} \to 1$,

$$\lim_{n \to \infty} \inf \int_0^T e^{g(t)} (\nabla F(t, u_\varepsilon(t))) \frac{u_\varepsilon}{\|u_\varepsilon\|} dt < 0,$$

and

$(F_4) - \infty < \lim\inf_{t \to \infty} \frac{\nabla F(t, x)}{|x|}$, uniformly;

for a.e. $t \in [0, T]$, then the problem (1.1) has at least one solution in $H^1_r$.

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Proof of Theorem 3.2 We divide the proof into several lemmas.

Lemma 3.1 If \((F_3)\) condition hold, then \(\varphi\) is anti-coercive on \(R^N\).

Proof First, we prove that there exist \(\delta > 0\),
\[
\rho > 0 \quad \text{such that} \quad \int_0^T \varphi'(t, x) \cdot x \, dt \leq -\delta |x| \quad \text{for all} \quad x \in R^N \quad \text{with} \quad |x| \geq \rho.
\]
If not, there is a sequence \(|x_n| \subset R^N \) with \(|x_n| \rightarrow \infty\) and
\[
\int_0^T \varphi'(t, x_n) \cdot \frac{x_n}{|x_n|} \, dt > \frac{1}{n}, \quad \forall n \geq 1.
\]
This contradicts \((F_3)\). Therefore, for \(u \in R^N\) with \(|u| > \rho\), one has
\[
\varphi(u) = \int_0^T \varphi'(t, u) \cdot u \, dt = \int_0^T \varphi'(t, u) \cdot \int_0^T (\nabla F(t, su), u) \, ds \, dt + \int_0^T \varphi'(t, 0) \cdot 0 \, dt + \int_0^T \varphi'(t, u) \cdot \int_0^T (\nabla F(t, su), u) \, ds \, dt + c_1,
\]
Since
\[
\left| \int_0^T \varphi'(t, u) \cdot \int_0^T (\nabla F(t, su), u) \, ds \, dt \right| \leq \int_0^T \varphi'(t, u) \cdot \int_0^T \left| \nabla F(t, su) \right| |u| \, ds \, dt \leq \int_0^T \rho |u| \varphi'(t, 0) \cdot b(t) \, dt \leq c_2,
\]
and
\[
\int_0^T \varphi'(t, u) \cdot \int_0^T (\nabla F(t, su), u) \, ds \, dt \leq -\delta |u| \left(1 - \frac{\rho}{|u|}\right) = -\delta |u| + \delta \rho,
\]
\[
\varphi(u) \rightarrow -\infty \quad \text{as} \quad |u| \rightarrow \infty, \quad \text{where} \quad a_\rho = \max_{|x| \leq \rho} \varphi(|x|).
\]

Lemma 3.2 Let \(H^1_T = \{u \in H^1_T: \int_0^T u(t) \, dt = 0\}\).

If condition \((F_3)\) holds, then \(\varphi\) is coercive on \(H^1_T\).

Proof By \((F_3)\) there exist \(\lambda < 0\) and \(M > 0\) such that \(\nabla F(t, x) \cdot x > \lambda |x| \) for all \(x \in R^N\) with \(|x| > M\) and a.e. \(t \in [0, T]\). Moreover, Let \(aM = \max_{|x| \leq M} \varphi(|x|)\). Then by \((A)\) we know that
\[
\left(\nabla F(t, x) \cdot x \right) \geq aM |x|, \quad \text{for all} \quad x \in R^N \quad \text{with} \quad |x| \leq M \quad \text{and a.e.} \quad t \in [0, T].
\]
Hence
\[
\left(\nabla F(t, x) \cdot x \right) \geq \lambda |x| - aM |x|,
\]
for all \(x \in R^N\) and a.e. \(t \in [0, T]\).

Consequently,
\[
F(t, x) = F(t, x) - F(t, 0) + F(t, 0) = \int_0^T \nabla F(t, sx) \cdot x \, ds + F(t, 0) \leq \lambda |x| - aM |x| + F(t, 0).
\]
If there are a constant \(c\) and a sequence \(|u_n| \subset H^1_T\) such that \(|u_n| \rightarrow \infty\) and \(\varphi(u_n) \leq c, n = 1, 2, \ldots\), then by Proposition 1.1 in [3], one has
\[
c \geq \varphi(u_n) = \frac{1}{2} \int_0^T \varphi'(t, u_n(t)) \cdot u_n(t) \, dt + \int_0^T \varphi'(t, u_n(t)) \cdot F(t, u_n(t)) \, dt \geq \frac{1}{2} \int_0^T \varphi'(t, u_n(t)) \cdot u_n(t) \, dt + \int_0^T \varphi'(t, u_n(t)) \cdot [\lambda |u_n(t)| - aM |u_n(t)|] \cdot u_n(t) + F(t, 0) \, dt \geq c_1 |u_n| - c_2 |u_n| - c_2,
\]
where \(c_1, c_2, c_3\) are positive constants. This contradicts that \(|u_n| \rightarrow 0\). Hence \(\varphi\) is coercive on \(H^1_T\).

Lemma 3.3 If a sequence \(|u_n| \subset H^1_T\) is such that \(\varphi'(un) \rightarrow 0\) and \(|u_n|\) is bounded in \(H^1_T\), then \(|u_n|\) has a convergent subsequence in \(H^1_T\).

Proof Since \(H^1_T\) is a Hilbert space, passing to a subsequence if necessary, we may assume that there is a point \(u_0 \in H^1_T\) such that
\[
u_n \rightarrow u_0 \quad \text{in} \quad H^1_T,
\]
and
\[
u_n \rightarrow u_0 \quad \text{in} \quad L^2([0, T]).
\]
By Proposition 1.2 in [3] we know that \(|u_n|\) converges uniformly to \(u_0\) on \([0, T]\). Hence there is a \(M > 0\) such that
\[
\max_{0 \leq t \leq T} |u_n(t)| \leq M, \quad n = 1, 2, \ldots
\]
Let \(aM = \max |x| \leq M \varphi'(x)\). Then by \((A)\) we know that \(|\nabla F(t; u_n(t))| \leq aM |t|\) for a.e. \(t \in [0, T]\).

Notice that
\[
(\varphi'(u_n) - \varphi'(u_0), u_n - u_0) = \int_0^T \varphi'(t, \nu_n(t)) \cdot \nu_n \, dt + \int_0^T \varphi'(t, \nu_n(t)) \cdot (\nabla F(t, u_n(t)) - \nabla F(t, u_0(t))) \cdot u_n(t) - u_0(t) \, dt.
\]
we have
\[
\int_0^T \varphi'(t, \nu_n(t)) \cdot \nu_n \, dt \leq \|\varphi'(u_n) - \varphi'(u_0)\| \cdot \|u_n - u_0\| + 2aM \|u_n - u_0\| \int_0^T \varphi'(t, b(t)) \, dt \rightarrow 0 \quad \text{as} \quad n, \quad m \rightarrow \infty, \quad \text{where} \quad \|u_n - u_m\| = \max_{0 \leq t \leq T} |u_n(t) - u_m(t)|.
\]
Consequently, \( \| u_n - u_m \|^\beta = \int_0^T e^{G(t)} \| \dot{v}_n(t) - \dot{v}_m(t) \|^\beta \, dt + \int_0^T e^{G(t)} \| u_n(t) - u_m(t) \|^\beta \, dt \to 0 \), as \( n, m \to \infty \), and hence \( \{ u_n \} \) is a Cauchy sequence in \( H^r_t \). By the completeness of \( H^r_t \), we know that \( \{ u_n \} \) is a convergent sequence in \( H^r_t \). This completes the proof.

**Lemma 3.4** If \( (F_3) \) condition and \( (F_4) \) hold, then \( \varphi \) satisfies the \((P.S.)\) condition. Proof. By the proof of Lemma 3.2, we know that there exist \( \lambda < 0 \) and \( M > 0 \) such that

\[
F(t, x) \geq \lambda \| x \| - aM^{(1)} \| x \| + F(t, 0).
\]

If a sequence \( \{ u_n \} \subset H^r_t \) is such that \( \varphi'(u_n) \to 0 \) and there exists a constant \( c \) such that \( \varphi(u_n) \leq c \), then the \( n = 1, 2, \cdots \), then \( \{ u_n \} \) is bounded in \( H^r_t \). Otherwise, passing to a subsequence if necessary, we may assume that \( \| u_n \| \to \infty \). Let \( v_n = \frac{u_n}{\| u_n \|} \). Since \( H^r_t \) is a Hilbert space, there is a point \( v_0 \in H^r_t \) and a sub-sequence of \( \{ v_n \} \), we still note by \( \{ v_n \} \), such that

\[
v_n \rightharpoonup v_0 \quad \text{in} \quad H^r_t,
\]

By Proposition 1.2 in [3] we know that \( \{ v_n \} \) converges uniformly to \( v_0 \) on \([0, T] \). Hence there is a \( M > 0 \) such that

\[
\max_{0 \leq t \leq T} \| u_n(t) \| \leq M, n = 1, 2, \cdots.
\]

Hence, by Proposition 1.1 in [3] one has

\[
\frac{c}{\| u_n \|} \leq \frac{\varphi'(u_n)}{\| v_n \|} = \frac{1}{2} \int_0^T e^{G(t)} \| \dot{v}_n(t) \|^\beta \, dt + \frac{1}{2} \int_0^T e^{G(t)} \| F(t, u_n(t)) \| \, dt \geq 0.
\]

\[
\frac{1}{2} \int_0^T e^{G(t)} \| \dot{v}_n(t) \|^\beta \, dt + \frac{1}{2} \int_0^T e^{G(t)} \| F(t, 0) \| \, dt = \frac{1}{2} - \frac{1}{2} \int_0^T e^{G(t)} \| v_n(t) \|^\beta \, dt - \frac{1}{2} \int_0^T e^{G(t)} \| v_n(t) \|^\beta \, dt - \frac{1}{2} \int_0^T e^{G(t)} \| F(t, 0) \| \, dt.
\]

Therefore, by \( (F_3) \) one has

\[
\liminf_{n \to \infty} \int_0^T e^{G(t)} \frac{\varphi(u_n)}{\| u_n \|} \, dt < 0,
\]

On the other hand,

\[
\int_0^T e^{G(t)} \left( \nabla F(t, u_n(t)) \right) \cdot \frac{u_n}{\| u_n \|} \, dt = \left( \varphi(u_n) \right) \left( \frac{u_n}{\| u_n \|} \right) \to 0 \quad \text{as} \quad n \to \infty.
\]

This is a contradiction. Hence \( \{ u_n \} \) is bounded in \( H^r_t \).

By virtue of Lemma 3.3, \( \{ u_n \} \) has a convergent subsequence in \( H^r_t \), and hence \( \varphi \) satisfies the \((P.S.)\) condition.

Sum up the above fact, Theorem 3.2 follows from Theorem 4.6 in [5].

**Remark 3.2** There are functions satisfying our Theorem 3.2. For example,

\[
F(t, x) = -|x|^\beta \sin \omega t (N \geq 2).
\]

References:


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