Bifurcations of Traveling Wave Solutions for a Class of Camassa-Holm Equations

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Abstract: By using the bifurcation theory of dynamical systems to a class of Camassa-Holm equation, the existence of traveling wave solutions is shown. Under different parametric conditions, explicit exact parametric representations of (1.1) are obtained.

Key words: solitary traveling wave solution; periodic traveling wave solution; smoothness of waves; Camassa-Holm equation


1 Introduction

The Camassa-Holm equation (C-H) equation is stated as:

\[ u_t - u_{txx} + uu_{xx} - \frac{1}{2} f(u)_x - ku = 0 \]  \hspace{1cm} (1.1)

Where \( f(u) = \alpha u^2 \) and \( k \) is a positive constant parameter.

Let \( u(x, t) = \varphi(x - ct) = \varphi(\xi) \), where \( c \) is wave speed. Substituting it into (1.1), integrating once and setting the constants of integration to be zero, then

\[ (\varphi - c)\varphi'' = \frac{1}{2} (\varphi')^2 + \frac{1}{2} f(\varphi) + (k - c)\varphi \]  \hspace{1cm} (1.2)

When \( \varphi'' \) is the derivative with respect to \( \xi \).

(1.2) is equivalent to the differential equation system

\[ \frac{d\varphi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\varphi''}{2(\varphi - c)} \]  \hspace{1cm} (1.3)

Which have the following first integral:

\[ H(\varphi, y) = (\varphi - c)y^2 - \frac{1}{3}(k - c)\varphi^3 + \frac{1}{3}\alpha \varphi^3 = h \]  \hspace{1cm} (1.4)

In this paper, we shall consider dynamical behavior of the traveling wave solutions of (1.1) for \( c > 0 \). It is noticed that the right hand of the second equation in (1.3) is not continuous when \( \varphi = c \). This phenomenon has been considered by some authors \( [3, 4] \).

The paper is organized as follows. In section 2, we discuss the bifurcation of phase portraits of (1.3). In section 3, we give all explicit exact parametric representations for traveling wave solutions of (1.1).
2. Bifurcations of phase portraits of (1.3)

Let \( \mathbf{\xi} = (\mathbf{\alpha} - c) \mathbf{\xi} \), then system (1.3) has the same topological phase portraits as the following polynomial system except for the straight line \( \mathbf{\alpha} = c \)

\[
\frac{d\mathbf{\xi}}{dt} = \mathbf{y} (\mathbf{\xi} - c),
\]

\[
\frac{d\mathbf{y}}{dt} = \frac{1}{2} [ - \mathbf{y}^2 + 2(k - \mathbf{c})\mathbf{\alpha} + \mathbf{c}] \tag{2.1}
\]

Denote that \( c = (k - c) \). Equation (2.1) has two equilibrium points at the origin \((0, 0)\) and \((\mathbf{\alpha}, 0)\)

\[
\begin{aligned}
\mathbf{J} (0, 0) &= \det M (0, 0) = c(k - c), & \mathbf{J} (c, \mathbf{\alpha}) &= \det M (c, \mathbf{\alpha}) = -Y_x^2 \\
\end{aligned}
\]

By the theory of planar dynamical system\(^{(3, 6)}\), for an equilibrium point of a planar Hamiltonian system, if \( J < 0 \) then the equilibrium point is a saddle point; if \( J > 0 \) then it is a center point; if \( J = 0 \) and the Poincaré index of the equilibrium point is 0, then this equilibrium point \((c, \mathbf{\alpha})\) is a cusp. We see from (2.2) that the equilibrium points are saddle points

We obtain the phase portraits of (2.1) as shown in Fig. 2.

3. The exact parametric representations of traveling wave solutions of (1.1)

In this section, we shall use the results in section 2 to obtain all periodic cusp wave solutions and all solitary cusp wave solutions

\[(1, 1) \text{ For } H(\mathbf{\alpha}, y) = h, \text{ in Fig } 2 (2 - 2), \text{ the smooth periodic wave solution have the parametric representation} \]

\[
u(x, t) = \mathbf{\alpha} (x - ct)
\]

\[
= - \frac{c}{4} - 1 + 3 \cos \left( \frac{\mathbf{\alpha}}{3} (x - ct) \right) \tag{3.1}
\]

\[(1, 2) \text{ For } H(\mathbf{\alpha}, y) = h, \text{ in Fig } 2 (2 - 3), \text{ two periodic cusp wave solutions have the parametric representation} \]

\[
u(x, t) = \mathbf{\alpha} (x - ct)
\]

\[
= \frac{1}{2} \left\{ (\mathbf{\alpha} - \mathbf{\alpha}^*) \cos \frac{2 \mathbf{\alpha} - \mathbf{\alpha}^*}{3} (x - ct) \right\}
\]

\[(3.2) \]

Where \( g_0 = \sqrt{3(k - c) + \mathbf{\alpha}} > 0, \mathbf{\alpha} = \frac{1}{2} (g_0 \pm \sqrt{g_0(g_0 + 4c)}) \) are the values of intersection points of the two arch curves with \( \mathbf{\alpha} \) - axis

\[(1, 3) \text{ For } H(\mathbf{\alpha}, y) = h, \text{ in Fig } 2 (2 - 5), \text{ the solitary cusp wave solution has the parametric representation} \]

\[
u(x, t) = \mathbf{\alpha} (x - ct)
\]

\[
= \left\{ (2 \mathbf{\alpha} - 1 + e^{-3(1 - e^{\mathbf{\alpha}} x - ct)}) \right\}
\]

\[(3.3) \]

\[(1, 4) \text{ For } H(\mathbf{\alpha}, y) = h, \text{ in Fig } 2 (2 - 6), \text{ the periodic cusp wave solution has the parametric representation} \]

\[
u(x, t) = \mathbf{\alpha} (x - ct)
\]

\[
= \left\{ (2 \mathbf{\alpha} + g_0) \cosh \frac{2 \mathbf{\alpha} + g_0}{3} (x - ct) \right\}
\]

\[(3.4) \]

Where \( \Delta_{\mathbf{\alpha}} = g_0 (g_0 - 4c) \)
For $H(\Phi, y) = h, y = 0$ in Fig. 2 (2 - 10), the solitary cusp wave solution has the parametric representation

$$u(x, t) = \Phi(x - ct)$$

$$0 \leq |x - ct| \leq \frac{3}{\sqrt{\alpha}} \ln \left( \frac{\alpha}{2(c - k)} \right) \quad (3.5)$$

**Fig. 2** The bifurcations of phase portraits of (2.1) for $\alpha \neq 0$

(2-1) $(\varepsilon, \alpha) \in (I_3), (2-2) (\varepsilon, \alpha) \in I_2, \alpha > c,$ (2-3) $(\varepsilon, \alpha) \in (I_4), (2-4) (\varepsilon, \alpha) \in (I_5), (2-5) (\varepsilon, \alpha) \in I_6, (2-6) (\varepsilon, \alpha) \in (I_7), (2-7) \varepsilon = 0, \alpha > 0,$ (2-8) $\varepsilon = 0, \alpha < 0,$ (2-9) $(\varepsilon, \alpha) \in (I_8), (2-10) \alpha = \frac{1}{4}(\varepsilon - k), k < c,$ (2-11) $(\varepsilon, \alpha) \in (I_9), (2-12) \alpha = \frac{1}{2}(\varepsilon - k), k < c,$ (2-13) $(\varepsilon, \alpha) \in (I_9), (2-14) (\varepsilon, \alpha) \in (I_9)$

References: