Almost Sure Limit Theorem for the Maximum and Minimum of Stationary Normal Vector Sequences

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Abstract: For a standardized stationary normal d dimensional random vectors sequence of \(X_1, X_2, \ldots\) we consider its maximum \(M_n\) and minimum \(m_n\). The almost sure central limit theorem for the maximum and minimum of stationary normal vector is proved under some suitable conditions.

Key words: almost sure limit theorem; stationary sequence; maximum and minimum

Introduction

Recently the almost sure limit central theorem (ASCLT) for the maximum of random variables which is discovered by Fahmer and Stadtmüller[1] and Cheng[2] et al. respectively. If there exist real sequence \(a_n > 0, b_n \in \mathbb{R}\) and a non-degenerate distribution \(G(x)\) such that \(P(M_n < a_n, X_n > b_n) \to (G)(x)\), then

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I(a_i(M_k - b_i) \leq x) = G(x) \quad \text{a.s.}
\]

(1)

for all \(x\), where \(M_n = \max_{1 \leq i \leq n} X_i\). Czakó and Gonchigdanzan[3] prove the ASCLT for the maximum of the stationary normal sequences under some weak dependent conditions. The research of Chen and Lin[4] is an extension of (1) to the non-stationary normal sequences. Maci"n Džudzink[5] concerns in the ASCLT for the maximum and sum of stationary normal sequence. Chen and Peng[6] consider the ASCLT of multivariate stationary normal sequences.

In our considerations, we will concentrate on the ASCLT for the maximum and minimum of the stationary normal sequences under some weak dependent conditions. Throughout this paper, \(X_1, X_2, \ldots\) is a standardized stationary Gaussian d dimensional vectors sequence satisfying

\[
EX_i = (EX_i(p) = 0, \ p = 1, \ldots, d), \ VarX_i = (EX_i(p) = 1, \ p = 1, \ldots, d), \ \eta_{ij}(p) = \text{Cov}(X_i(p), X_j(p)) = \eta_{ij}(p), \ \eta_{ij}(p, q) = \text{Cov}(X_i(p), X_j(q)) = \eta_{ij}(p, q), \ M_n = (M_n(1), \ldots, (M_n(d)) ), \ M_n(p) = \max_{1 \leq i \leq n} X_i(p), \ 1 \leq p \neq q \leq d, \]

\[ \eta_{ij}(p) = \frac{\eta_{ij}(p, q)}{\eta_{ij}(q)} \]

and

\[ \eta_{ij}(p, q) = \frac{\eta_{ij}(p, q)}{\eta_{ij}(q)} \]

for all \(p, q\).
and $u_0 = (u_0(1), \ldots, u_0(d))$ is a real vector, $u_0 > u_0$ denotes $u_0(p) > u_0(p)$, $p = 1, \ldots, d$ and $a \leq b$ stands for $a = O (a b)$. $a_0 = \sqrt{2 \log n}, b_0 = a_0 - \frac{1}{2} a_0^{-1} \log (\log n)$.

1 Main Results

**Theorem 1.1** Let $X_1, X_2, \ldots$ be a standardized stationary normal $d$ dimensional vectors sequence satisfying

(a) $\xi_n(p) \to 0$, $\xi_n(p,q) \to 0$ for $1 \leq p \neq q \leq d$ as $n \to \infty$, and,

(b) For some $\varepsilon > 0$ and $\gamma \geq \frac{2(1 + \delta)}{1 - \delta}$ with $\delta = \max \{\mathbb{E} \sup_{p \leq 0} | \xi_n(p) |, \mathbb{E} \sup_{p \leq 0} | \xi_n(p,q) | \} < 1$, satisfying

$$\frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} (| \xi_n(p) \log k | \exp (| \xi_n(p) | \log k) \leq (\log n)^{-(1+\varepsilon)}$$

$$\frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} (| \xi_n(p,q) \log k | \exp (| \xi_n(p,q) | \log k) \leq (\log n)^{-(1+\varepsilon)}$$

(1) If $n(1 - \Phi (u_0(p))) \to \tau$, $\Phi (v_n(p)) \to \eta$, for $0 \leq \tau, \eta < \infty$, $p = 1, \ldots, d$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} I(| \xi_n(p) \log k | \exp (| \xi_n(p) | \log k) \to \eta, \sigma_k \to \mu_k$$

(2) If $u_0(p) = a_0^{-1} x_p + b_p$, $v_n(p) = - a_0^{-1} y_p - b_p$, $x_p$, $y_p$ is real number, $p = 1, \ldots, d$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} I(| \xi_n(p) \log k | \exp (| \xi_n(p) | \log k) \to \eta, \sigma_k \to \mu_k$$

**Corollary 1.2** Under condition of Theorem 1.1, if $n(1 - \Phi (u_0(p))) \to \tau$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} I(| \xi_n(p) \log k | \exp (| \xi_n(p) | \log k) \to \eta, \sigma_k \to \mu_k$$

If $u_0(p) = a_0^{-1} x_p + b_p$, for $p = 1, \ldots, d$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{p=1}^{d} \sum_{k=1}^{n} I(| \xi_n(p) \log k | \exp (| \xi_n(p) | \log k) \to \eta, \sigma_k \to \mu_k$$

**Corollary 1.3** Let $X_1, X_2, \ldots$ be a standardized stationary normal $d$ dimensional vectors sequence satisfying

$$\xi_n(p) \log n(\log n)^{-\gamma} = O(1), \xi_n(p,q) \log n(\log n)^{-\gamma} = O(1)$$

If $n(1 - \Phi (u_0(p))) \to \tau$, $\Phi (v_n(p)) \to \eta$, for $0 \leq \tau, \eta < \infty$ and $\varepsilon > 0$, then (2) holds; especially as $u_0(p) = a_0^{-1} x_p + b_p$, $v_n(p) = - a_0^{-1} y_p - b_p$, $x_p$, $y_p \in \mathbb{R}$, $p = 1, \ldots, d$, then (3) holds.

2 Proofs

**Lemma 2.1** Let $(\xi_n, \eta_n)_n$ be $d$ dimensional standardized stationary normal sequences with $\xi_n(p) = \text{Cov} \xi_n(p), \eta_n(p)$, $\xi_n(p,q) = \text{Cov} \xi_n(p), \eta_n(q)$; $\text{Cov} \xi_n(p), \eta_n(p)$, $\text{Cov} \xi_n(p,q), \eta_n(p)$, $\text{Cov} \eta_n(p,q), \eta_n(q)$. Denote $\rho(p) = \max (| \rho_0(p) |), | \rho_0(p, q) |, | \rho_0(p, q) |$ and let $u_1, \ldots, u_d$ be real numbers. If $\max (| \rho_0(p) |) < 1$, $\max (| \rho_0(p, q) |) < 1$ for $p = 1, \ldots, d$ with $\omega_0 = \min (| v_0(1) \ldots |, | v_0(d) |)$, then

$$| P(\eta_n | u_n(1) \ldots | u_n(d) |) |$$

$$\leq K_1 \sum_{p=1}^{d} \sum_{1 \leq p \neq q \leq d} | \xi_n(p, q) - \xi_n(p) | (1 - \rho_0^2(p, q))^{1/2} \exp \left( - \frac{\omega_0^2}{1 + \rho_0^2(p, q)} \right)$$

where $K_1, K_2$ are absolute constants.

**Proof** The proof is similar to Theorem 4.3.2 in Leadbetter[1], etc.

**Lemma 2.2** Let $X_1, X_2, \ldots$ be a sequence of standardized stationary normal $d$ dimensional vectors such that

(a) (b) of Theorem 2.1 hold and $n(1 - \Phi (u_0(p)))$ and $n(\Phi (v_n(p)))$ are bounded for some $\varepsilon > 0$ as $p = 1, \ldots, d$. 205
d, then
\[
\sup_{1 \leq i \leq n} \sum_{p=1}^{n} \sum_{r=0}^{n} | \xi_i(p, q) | \left\{ \exp \left[ -\frac{\omega_i^2(p) + \omega_i^2(q)}{2(1 + |r_i(p, q)|)} \right] + \exp \left[ -\frac{\nu_i^2(p) + \nu_i^2(q)}{2(1 + |r_i(p, q)|)} \right] \right\} \ll (\log \log n)^{-1/6}.
\]

Proof The front part of (4) can be controlled by
\[
\sup_{1 \leq i \leq n} \sum_{p=1}^{n} \sum_{r=0}^{n} | r_i(p, q) | \left\{ \exp \left[ -\frac{\omega_i^2(p) + \omega_i^2(q)}{2(1 + |r_i(p, q)|)} \right] + \exp \left[ -\frac{\nu_i^2(p) + \nu_i^2(q)}{2(1 + |r_i(p, q)|)} \right] \right\} \ll (\log \log n)^{-1/6},
\]
where \( C \) is a constant. By assumption \( n(1 - \Phi (\mu_n)) \) and \( n(1 - \Phi (\nu_n)) \) are bounded, thereby \( n(1 - \Phi (\omega_n)) \leq K \), \( p = 1, \ldots, d \). Let \( \mu_n \equiv \omega_n \) if \( n \leq K \) and \( n(1 - \Phi (\omega_n)) = K \) if \( n > K \) Then clearly \( \omega_n \geq \mu_n \).

By virtue of the fact \( \Phi(x) \) is the standard normal density function, we can get
\[
\exp \left( -\frac{\mu_i^2}{2} \right) \frac{K}{\sqrt{n}} \ll \sqrt{\log n}, \quad p = 1, \ldots, d
\]
(6)

Define \( \beta \) to be 0. < 2

Note that
\[
k \sum_{i \neq j, i, j \in [n]} \exp \left( -\frac{\mu_i^2 + \mu_j^2}{2(1 + |r_i(p, q)|)} \right) \leq k \sum_{i \neq j, i, j \in [n]} \exp \left( -\frac{\mu_i^2 + \mu_j^2}{2(1 + |r_i(p, q)|)} \right) = A_1 + B_1
\]
Using (6)
\[
A_1 \leq k \sum_{i \neq j, i, j \in [n]} n \exp \left[ -\frac{\omega_i^2 + \omega_j^2}{2(1 + |r_i(p, q)|)} \right] \leq \sum_{i \neq j, i, j \in [n]} k n \left( \frac{\log k \log n}{kn} \right)^{1/6}
\]
\[
\leq \sum_{i \neq j, i, j \in [n]} n \beta \frac{1}{(\log n)^{1/6}}
\]
Since \( 1 + \beta + \frac{2}{1 + \delta} < 0 \), we have \( A_1 \ll (\log \log n)^{-1/6} \).

\[
B_1 \ll \sum_{i \neq j, i, j \in [n]} k n \sum_{i \neq j, i, j \in [n]} \xi_i(p, q) \left( \frac{\log k \log n}{kn} \right)^{1/6} \leq \frac{1}{\beta} \frac{1}{n} \sum_{i \neq j, i, j \in [n]} \xi_i(p, q) \left( \frac{\log k \log n}{kn} \right)^{1/6} \leq \frac{1}{\beta} \frac{1}{n} \sum_{i \neq j, i, j \in [n]} \Xi_i(p, q) \left( \frac{\log k \log n}{kn} \right)^{1/6} \leq (\log \log n)^{-1/6}
\]
(6)

is proved and (5) is similar

Lemma 2.3 Let \( X_1, X_2, \ldots \) be a standardized stationary normal \( d \) dimensional vectors sequence satisfying
(a) \( \xi_i(p, q) \to 0 \), \( \xi_i(p, q) \to 0 \) for \( 1 \leq p \neq q \leq d \) as \( n \to \infty \).
(b) For some \( \delta > 0 \) and \( \gamma \geq 2(1 + \delta) \)
\[
\frac{1}{\beta} \frac{1}{n} \sum_{i \neq j, i, j \in [n]} \xi_i(p, q) \left( \frac{\log k \log n}{kn} \right)^{1/6} \to 0,
\]
(1) if \( n(1 - \Phi (\mu_n)) \to \Xi, n(1 - \Phi (\nu_n)) \to \Xi, \) for \( 0 \leq \Xi, \Xi \leq \infty \), \( p = 1, \ldots, d \), then
By the basic inequality $x^{n-k} - x^n \leq \frac{k}{n}$ with $0 \leq x \leq 1$, we have

$$E \left| I \left[ v_i < m_a \leq M_a \leq u_a \right] - I \left[ v_i < m_{k_a} \leq M_{k_a} \leq u_a \right] \right| \leq \frac{k}{n} + \left( \log \log n \right)^{-(1+\varepsilon)}.$$
By Lemma 2.1 and Lemma 2.2, we have

$$\left| \text{Cov}(I/M_1 \leq u_k, m_k > v_k), I/M_{k+n} \leq u_k, m_{k+n} > v_k) \right| \leq (\log \log n)^{-\left(1 - \frac{1}{\varepsilon}\right)}.$$

**Proof** By Lemma 2.1 and Lemma 2.2, we have

$$P(v_k < m_k \leq M_k \leq u_k) \leq \sum_{i=1}^{n} \frac{1}{k} \sum_{j=1}^{n} \text{Var}(\alpha_{i,j}) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(\alpha_{i,j}, \alpha_{i,j}) = D_1 + D_2.$$

$$\left| \text{Cov}(I(v_k < m_k \leq M_k \leq u_k), I(v_k < m_k \leq M_k \leq u_k)) \right| \leq \left| \text{Cov}(I(v_k < m_k \leq M_k \leq u_k), I(v_k < m_k \leq M_k \leq u_k)) \right| +$$

$$\left| \text{Cov}(I(v_k < m_k \leq M_k \leq u_k), I(v_k < m_k \leq M_k \leq u_k)) \right| \equiv E_1 + E_2.$$

By Lemma 2.4 and Lemma 2.5, we have $E_1 \ll \left(\frac{k}{l}\right) + (\log \log l)^{-\left(1 - \frac{1}{\varepsilon}\right)}$, $E_2 \ll (\log \log l)^{-\left(1 - \frac{1}{\varepsilon}\right)}$. Hence,

$$D_2 \ll \sum_{i=1}^{n} \frac{1}{k} \sum_{j=1}^{n} \frac{1}{k l} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{k l} (\log \log l)^{-\left(1 - \frac{1}{\varepsilon}\right)} \leq$$

$$\sum_{i=1}^{n} \frac{1}{k l} + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{1}{k l} (\log \log n)^{-\left(1 - \frac{1}{\varepsilon}\right)} \leq$$

$$\log n + \log^2 n (\log \log l)^{-\left(1 - \frac{1}{\varepsilon}\right)}.$$

By Lemma 2.1 in Csáki and Gonchigdanzan \(3^{(1)}\) and Lemma 2.3, Theorem 1.1 is proved.

**Proof of Corollary 1.3** The condition of Corollary 1.3 is weaker than Theorem 1.1, it is easily proved.

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References:


